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ON THE LAPLACE-STIELTJES TRANSFORMATIONS OF SOME PROBABILITY DISTRIBUTIONS

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1 Introduction

It is known in [8] that all of probability distributions with density of normed product of the Cauchy densities such as

$$f(a,b;x)=\frac{c}{(a^2+x^2)(b^2+x^2)},\ (0<a<b)$$

are infinitely divisible. But it seems that it is not known whether a probability distriution with density of normed product of the multidimensional Cauchy densities is infinitely divisible or not. In this paper we will show the infinite divisiblity of some probability distributions with density of normed product of the 3 dimensional Cauchy densities, namely,

$$f(a,b;x)=\frac{c}{(a^2+|x|^2)^{(d+1)/2}(b^2+|x|^2)^{(d+1)/2}},$$

where c is a normalised constant and

$$0<a<b;\ d=3;\ x=(x_1,x_2,x_3)\in\mathbf{R}^3.$$

We assume the dimension d is an odd integer since $(d+1)/2$ is an integer, but it should be noted that the density $f(a,b;x)$ can not be decomposed to a sum of partial fractions in the same way as in the 1 dimensional case. In this paper we overcomed this difficulty. A probability distribution function $F(x)$ is called an infinitely divisible probability distribution if for each integer $n>1$ there is a probability distribution $F_n(x)$ such that

$$F(x)=(F_n*\cdots*F_n)(x),$$

(* denotes the convolution.). If a probability distribution function $F(x)$ is 0 on the interval $(-\infty, 0)$ and infinitely divisible and if we denote the Laplace-Stieltjes taransforms of the probability distributions $F(x)$ and $F_n(x)$,

$$\zeta(s) = \int_0^\infty e^{-sx} dF(x), \quad \zeta_n(s) = \int_0^\infty e^{-sx} dF_n(x),$$

the equality

$$\zeta(s) = (\zeta_n(s))^n$$

holds. It is known that the Laplace-Stieltjes transform of an infinitely divisible probability distribution $F(x)$ on $[0, \infty)$ can be written as follows:

$$\zeta(s) = \exp\{\int_{-0}^\infty (e^{-sx} - 1)\frac{1}{x} dK(x)\}$$

where

(c1) $K(x)$ is nondecreasing,

(c2) $K(-0) = 0$,

(c3) $\int_1^\infty 1/x \, dK(x) < \infty$.

An infinitely divisible probability distribution $F(x)$ on $[0, \infty)$ satisfies an integral equation,

$$\int_0^x t \, dF(t) = \int_0^x F(x-t) dK(t), \; x > 0$$

and in particular if the probability distribution function $F(x)$ on $[0, \infty)$ has a density function $f(x)$, the density funcion $f(x)$ satisfies the following integral equation:

$$x f(x) = \int_0^x f(x-t) dK(t), \; x > 0.$$

(cf. F. Steutel [7])

2 Normed product of Cauchy densities

In this section and the following section, assume that

$$0 < a < b; \; d = 1, \, 3, \, 5, \, \cdots; x = (x_1, x_2, \cdots, x_d) \in \mathbf{R}^d.$$

Let us consider the following probability density which is consisting of normed product of the d dimensional Cauchy densities. That is

$$f(a,b;x) = \frac{c}{(a^2+|x|^2)^{(d+1)/2}(b^2+|x|^2)^{(d+1)/2}}, \tag{1}$$

where c is a normalised constant. It holds that

$$\frac{\Gamma((d+1)/2)}{(a^2+|x|^2)^{(d+1)/2}} = \int_0^\infty \exp\{-t(a^2+|x|^2)\}\cdot t^{(d+1)/2-1}dt.$$

From this we can write the mixture density as the following form

$$\begin{aligned} & \frac{\{\Gamma((d+1)/2)\}^2}{(a^2+|x|^2)^{(d+1)/2}(b^2+|x|^2)^{(d+1)/2}} \\ = & \int\int_{0\le t<\infty, 0\le u<\infty} \exp\{-(t+u)|x|^2 - a^2t - b^2u\} \\ & \cdot (tu)^{(d+1)/2-1}dtdu. \end{aligned} \tag{2}$$

Therefore we obtain a characteristic functions in which a normalised constant is ignored.

$$\begin{aligned} & \int_{R^d} \exp(izx)\frac{\{\Gamma((d+1)/2)\}^2}{(a^2+|x|^2)^{(d+1)/2}(b^2+|x|^2)^{(d+1)/2}}dx \\ = & \int_0^\infty\int_0^\infty \exp\{-a^2t-b^2u\}\cdot(tu)^{(d+1)/2-1}dtdu \\ & \cdot\int_{R^d}\exp\{izx-(t+u)|x|^2\}dx \\ = & \int_0^\infty\int_0^\infty (\frac{\pi}{t+u})^{d/2}\exp\{-|z|^2/(4(t+u))-a^2t-b^2u\} \\ & \cdot(tu)^{(d+1)/2-1}dtdu. \end{aligned} \tag{3}$$

By a change of variables,

$$t+u=v,\ t=t,$$

we see that

$$\begin{aligned} (3) = & \int_0^\infty\int_{(0,v]}(\frac{\pi}{v})^{d/2}\exp\{-\frac{|x|^2}{4v}\} \\ & \cdot\exp\{-a^2t-b^2(v-t)\}\cdot t^{(d+1)/2-1}(v-t)^{(d+1)/2-1}dtdv \\ = & \int_0^\infty \pi^{d/2}v^{d/2}\exp\{-\frac{|z|^2}{4v}-b^2v\}dv \\ & \cdot\int_0^1\exp\{(b^2-a^2)vy\}\cdot y^{(d+1)/2-1}(1-y)^{(d+1)/2-1}dy, \end{aligned} \tag{4}$$

Again, by a change of variable, $v = 1/u$, we have

$$(4) = \int_0^\infty \exp\{-\frac{u|z|^2}{4}\} \cdot \frac{\pi^{d/2}}{u^{d/2+2}} \exp\{-\frac{b^2}{u}\} \int_0^1 \exp\{\frac{(b^2-a^2)y}{u}\}$$
$$\cdot y^{(d+1)/2-1}(1-y)^{(d+1)/2-1} dy. \tag{5}$$

We can rewrite $f(a, b; x)$ such as the following form

$$\begin{aligned} f(a,b;x) &= \int_0^\infty \frac{1}{(\pi u)^{d/2}} \exp\{-\frac{|x|^2}{u}\} du \\ &\quad \cdot \frac{c\pi^{d/2}}{u^{d/2+2}} \cdot \exp\{-\frac{b^2}{u}\} \int_0^1 \exp\{\frac{(b^2-a^2)y}{u}\} \\ &\quad \cdot y^{(d+1)/2-1}(1-y)^{(d+1)/2-1} dy. \end{aligned} \tag{6}$$

Let us denote

$$\begin{aligned} g(d;u) := \frac{c\pi^{d/2}}{u^{d/2+2}} \cdot \exp\{-\frac{b^2}{u}\} \int_0^1 \exp\{\frac{(b^2-a^2)v}{u}\} \\ \cdot v^{(d+1)/2-1}(1-v)^{(d+1)/2-1} dv. \end{aligned} \tag{7}$$

The density function $g(d; u)$ is a mixing density of the d dimensional normal distributions. For the case $d = 1$ we have

$$g(1;u) = \frac{c\pi^{1/2}}{(b^2-a^2)u^{3/2}}(\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\}).$$

For the cases $d = 3,\ 5,\ 7,\ 9$ we obtain

$$\begin{aligned} g(3;u) &= \frac{c\pi^{3/2}}{(b^2-a^2)^2}\Big[\frac{1}{u^{3/2}}(\exp\{-\frac{a^2}{u}\} + \exp\{-\frac{b^2}{u}\}) \\ &\quad - \frac{2}{(b^2-a^2)u^{1/2}}(\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\})\Big], \end{aligned} \tag{8}$$

$$\begin{aligned} g(5;u) &= \frac{c\pi^{5/2}}{(b^2-a^2)^3}\Big[\frac{2!}{u^{3/2}}(\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\}) \\ &\quad - \frac{2\cdot 3!}{(b^2-a^2)u^{1/2}}(\exp\{-\frac{a^2}{u}\} + \exp\{-\frac{b^2}{u}\}) \\ &\quad + \frac{4!u^{1/2}}{(b^2-a^2)^2}(\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\})\Big], \end{aligned} \tag{9}$$

$$\begin{aligned} g(7;u) &= \frac{c\pi^{7/2}}{(b^2-a^2)^4}\Big[\frac{3!}{u^{3/2}}(\exp\{-\frac{a^2}{u}\}+\exp\{-\frac{b^2}{u}\}) \\ &-\frac{3\cdot 4!}{(b^2-a^2)u^{1/2}}(\exp\{-\frac{a^2}{u}\}-\exp\{-\frac{b^2}{u}\}) \\ &+\frac{3\cdot 5!u^{1/2}}{(b^2-a^2)^2}(\exp\{-\frac{a^2}{u}\}+\exp\{-\frac{b^2}{u}\}) \\ &-\frac{6!u^{3/2}}{(b^2-a^2)^3}(\exp\{-\frac{a^2}{u}\}-\exp\{-\frac{b^2}{u}\})\Big], \end{aligned} \tag{10}$$

$$\begin{aligned} g(9;u) &= \frac{c\pi^{9/2}}{(b^2-a^2)^5}\Big[\frac{4!}{u^{3/2}}(\exp\{-\frac{a^2}{u}\}-\exp\{-\frac{b^2}{u}\}) \\ &-\frac{4\cdot 5!}{(b^2-a^2)u^{1/2}}(\exp\{-\frac{a^2}{u}\}+\exp\{-\frac{b^2}{u}\}) \\ &+\frac{6\cdot 6!u^{1/2}}{(b^2-a^2)^2}(\exp\{-\frac{a^2}{u}\}-\exp\{-\frac{b^2}{u}\}) \\ &-\frac{4\cdot 7!u^{3/2}}{(b^2-a^2)^3}(\exp\{-\frac{a^2}{u}\}+\exp\{-\frac{b^2}{u}\}) \\ &+\frac{8!u^{5/2}}{(b^2-a^2)^4}(\exp\{-\frac{a^2}{u}\}-\exp\{-\frac{b^2}{u}\})\Big]. \end{aligned} \tag{11}$$

For the general dimension $d = 2l+1$ ($l = 5, 6, ...$) we obtain a formula

$$\begin{aligned} &g(d;u) = \frac{c\pi^{d/2}}{(b^2-a^2)^{l+1}} \\ &\cdot\Sigma_{j=0}^{l}\frac{(-1)^{2l-j}(l+j)!}{(a^2-b^2)^j u^{3/2-j}}\binom{l}{j}(\exp\{-\frac{a^2}{u}\}+(-1)^{l+1+j}\exp\{-\frac{b^2}{u}\}). \end{aligned} \tag{12}$$

3 Laplace-Stieltjes transformations of mixing densities

Let us denote the Laplace-Stieltjes transformation of the mixing density $g(d;u)$ by $\zeta(d;s)$ and let us take $\zeta(d;+0) = 1$. We will make use of the polar coordinate

$$s = re^{i\theta}, \ (-\pi < \theta < \pi, 0 < r).$$

Then

$$\sqrt{s} = \sqrt{r}e^{i\theta/2} = \sqrt{r}(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2})$$

and $0 < \sqrt{r}\cos\theta/2$. For the case $d = 1$ then

$$\begin{aligned}\zeta(1;s) &= \int_0^\infty \exp\{-su\}g(1;u)du \\ &= \frac{c\pi}{(b^2-a^2)}(\frac{1}{a}\exp\{-2a\sqrt{s}\} - \frac{1}{b}\exp\{-2b\sqrt{s}\}) \qquad (13)\end{aligned}$$

For the case $d = 3$ we have

$$\begin{aligned}\zeta(3;s) &= \frac{c\pi^2}{(b^2-a^2)^2}\Big\{(\frac{1}{a}\exp\{-2a\sqrt{s}\} + \frac{1}{b}\exp\{-2b\sqrt{s}\}) \\ &\quad -\frac{2}{(b^2-a^2)\sqrt{s}}(\exp\{-2a\sqrt{s}\} - \exp\{-2b\sqrt{s}\})\Big\}, \qquad (14)\end{aligned}$$

and for the case $d = 5$ we have

$$\begin{aligned}\zeta(5;s) &= \frac{c\pi^3}{(b^2-a^2)^3}[2!(\frac{1}{a}\exp\{-2a\sqrt{s}\} - \frac{1}{b}\exp\{-2b\sqrt{s}\}) \\ &\quad -\frac{2\cdot 3!}{(b^2-a^2)\sqrt{s}}(\exp\{-2a\sqrt{s}\} + \exp\{-2b\sqrt{s}\}) \\ &\quad +\frac{4!}{(b^2-a^2)^2 s}(a\ \exp\{-2a\sqrt{s}\}(1+\frac{1}{2a\sqrt{s}}) \\ &\quad -b\ \exp\{-2b\sqrt{s}\}(1+\frac{1}{2b\sqrt{s}})\Big\}. \qquad (15)\end{aligned}$$

Making use of the formula

$$K_\nu(z) = \frac{1}{2}(\frac{1}{2}z)^\nu \int_0^\infty \exp\{-t - \frac{z^2}{4t}\}\frac{dt}{t^{\nu+1}}, \qquad (16)$$

we obtain a Laplace-Stieltjes transformation for the general case,

$$\begin{aligned}\zeta(d;s) = \frac{c\pi^{d/2}}{\{(b^2-a^2)^{l+1}}\Sigma_{j=0}^{l}\frac{(-1)^{2l-j}(l+j)!}{(b^2-a^2)^j}\binom{l}{j}2^{j+1/2}\,. \\ \Big\{a^{2j-1}\frac{K_{(j-1)+1/2}(2a\sqrt{s})}{(2a\sqrt{s})^{j-1+1/2}} + (-1)^{l+1+j}b^{2j-1}\frac{K_{(j-1)+1/2}(2b\sqrt{s})}{(2b\sqrt{s})^{j-1+1/2}}\Big\}, \quad (17)\end{aligned}$$

where

$$K_{n+1/2}(z) = (\frac{\pi}{2z})^{1/2}\exp\{-z\}\Sigma_{r=0}^{n}\frac{(n+r)!}{r!(n-r)!(2z)^r} \qquad (18)$$

(cf. G. Watson [10]).

4 Laplace-Stieltjes transformations without zeros on the right half plane except for the origin

In this section we will show that some 3 dimensional probability distributions are infinitely divisible. For the case $d = 3$ we obtained the Laplace-Stieltjes transformation

$$\begin{aligned}\zeta(3;s) &= c_1\Big\{(b^2-a^2)(\frac{1}{a}\exp\{-2a\sqrt{s}\}+\frac{1}{b}\exp\{-2b\sqrt{s}\})\\ &\quad -\frac{2}{\sqrt{s}}(\exp\{-2a\sqrt{s}\}-\exp\{-2b\sqrt{s}\})\Big\},\end{aligned}$$

where we let $c_1 = c\pi^2/(b^2-a^2)^3$. From the above we have

$$\begin{aligned}&\zeta'(3;s) = -c_1\frac{1}{\sqrt{s}}\Big\{(b^2-a^2)(\exp\{-2a\sqrt{s}\}+\exp\{-2b\sqrt{s}\})\\ &-\frac{2}{\sqrt{s}}(a\ \exp\{-2a\sqrt{s}\}-b\ \exp\{-2b\sqrt{s}\})\\ &-\frac{1}{s}(\exp\{-2a\sqrt{s}\}-\exp\{-2b\sqrt{s}\})\Big\}. \qquad (19)\end{aligned}$$

On a neighborhood at the origin the function $\zeta(3;s)$ does not vanish. In fact, we have

$$\begin{aligned}&\zeta(3;s) = c_1\Big\{(b^2-a^2)(\frac{1}{a}(1-2a\sqrt{s}+\frac{(2a\sqrt{s})^2}{2!}+0(s^{3/2}))\\ &-\frac{1}{b}(1-2b\sqrt{s}+\frac{(2b\sqrt{s})^2}{2!}+0(s^{3/2})))\\ &-\frac{2}{\sqrt{s}}((1-2a\sqrt{s}+\frac{(2a\sqrt{s})^2}{2!}-\frac{(2a\sqrt{s})^3}{3!}+0(s^2))\\ &-(1-2b\sqrt{s}+\frac{(2b\sqrt{s})^2}{2!}-\frac{(2b\sqrt{s})^3}{3!}+0(s^2))\Big\}\\ =\ & c_1\Big\{(b-a)(\frac{(b+a)^2}{ab}-4)\\ &+2((b^2-a^2)(a+b)-\frac{4}{3}(b^3-a^3))s+0(s^{3/2})\Big\} \qquad (20)\end{aligned}$$

and see that $\zeta(3;s) \neq 0$ at a neighborhood of the origin since $4ab < (a+b)^2$. From the calculation of the following

$$\zeta'(3;s) = -\frac{c_1}{\sqrt{s}}\Big\{(b^2-a^2)((1-2a\sqrt{s}+\frac{(2a\sqrt{s})^2}{2!}+0(s^{3/2}))$$

$$
\begin{aligned}
&-(1-2b\sqrt{s}+\frac{(2b\sqrt{s})^2}{2!}+0(s^{3/2})))\\
&-\frac{2}{\sqrt{s}}(a(1-2a\sqrt{s}+\frac{(2a\sqrt{s})^2}{2!}+0(s^{3/2}))\\
&-b(1-2b\sqrt{s}+\frac{(2b\sqrt{s})^2}{2!}+0(s^{3/2}))\\
&-\frac{1}{s}((1-2a\sqrt{s}+\frac{(2a\sqrt{s})^2}{2!}-\frac{(2a\sqrt{s})^3}{3!}+0(s^2))\\
&-(1-2b\sqrt{s}+\frac{(2b\sqrt{s})^2}{2!}-\frac{(2b\sqrt{s})^3}{3!}+0(s^2)))\}\\
&=-c_1\{\frac{2}{3}(b-a)^3+0(s^{1/2})\} \qquad (21)
\end{aligned}
$$

we see that $\zeta'(3;s)$ is bounded at a neighborhood of the origin.

Theorem 1. *Assume that an inequality*

$$0<b^3-b^2a-ba^2-a^3$$

holds. Then the Laplace-Stieltjes transformation $\zeta(3;s)$ does not vanish on the right half complex plane except for the origin.

Proof. Let us denote $\sqrt{s}$ by z. Let us denote

$$
\begin{aligned}
&p(z)=\zeta(3;s)\sqrt{s}\exp\{2a\sqrt{s}\}/\{c_1\}\\
=\;&(b^2-a^2)(\frac{1}{a}+\frac{1}{b}\exp\{-2(b-a)z\})z+2(\exp\{-2(b-a)z\}-1). \qquad (22)
\end{aligned}
$$

We will show that $\Im p(z)\neq 0$ on the right half plane except for the origin. Making use of the polar coordinate

$$z=r\exp\{it\}=r\cos t+r\sin t,\quad (0\leq t\leq \pi/2)$$

we see that

$$
\begin{aligned}
&p(r\exp\{it\})\\
=\;&(b^2-a^2)(\frac{1}{a}+\frac{1}{b}\exp\{-2(b-a)(r\cos t+ir\sin t)\})\\
&\cdot(r\cos t+ir\sin t)\\
&+2(\exp\{-2(b-a)(r\cos t+ir\sin t)\}-1)\\
=\;&(b^2-a^2)(\frac{1}{a}+\frac{1}{b}\exp\{-2(b-a)r\cos t\}\cos(2(b-a)r\sin t)\\
&-i\frac{1}{b}\exp\{-2(b-a)r\cos t\}\sin(2(b-a)r\sin t))(r\cos t+ir\sin t)\\
&+2(\exp\{-2(b-a)r\cos t\}\cos(2(b-a)r\sin t)-1\\
&-i\exp\{-2(b-a)r\cos t\}\sin(2(b-a)r\sin t)). \qquad (23)
\end{aligned}
$$

Let us denote

$$\begin{aligned}
& \Im p(r\exp\{it\}) \\
= & (b^2-a^2)\frac{1}{a}r\sin t \\
& +(b^2-a^2)\frac{1}{b}\exp\{-2(b-a)r\cos t\}\cos(2(b-a)r\sin t)r\sin t \\
& -(b^2-a^2)\frac{1}{b}\exp\{-2(b-a)r\cos t\}\sin(2(b-a)r\sin t)r\cos t \\
& -2\exp\{-2(b-a)r\cos t\}\sin(2(b-a)r\sin t). \qquad (24)
\end{aligned}$$

When $t=0$ the Laplace transformation $\zeta(3;s)$ is positive. We take $0<t\leq\pi/2$. We see that

$$\begin{aligned}
& \Im p(r\exp\{it\}) \\
= & (b^2-a^2)\frac{1}{a}r\sin t-(b^2-a^2)\frac{1}{b}r\sin t\ \exp\{-2(b-a)r\cos t\} \\
& +\exp\{-2(b-a)r\cos t\}[2(b^2-a^2)\frac{r\sin t}{b}\cos^2((b-a)r\sin t) \\
& -2\{(b^2-a^2)\frac{r\cos t}{b}+2\}\cos((b-a)r\sin t)\sin((b-a)r\sin t)] \\
= & \exp\{-2(b-a)r\cos t\}2(b^2-a^2)\frac{r\sin t}{b}[\cos((b-a)r\sin t) \\
& -\frac{(b^2-a^2)r\cos t+2b}{2(b^2-a^2)r\sin t}\sin((b-a)r\sin t)]^2+R, \qquad (25)
\end{aligned}$$

where we let

$$\begin{aligned}
R = & (b^2-a^2)\frac{1}{a}r\sin t-(b^2-a^2)\frac{1}{b}r\sin t\ \exp\{-2(b-a)r\cos t\} \\
& -\exp\{-2(b-a)r\cos t\}2(b^2-a^2)\frac{r\sin t}{b} \\
& \cdot[\frac{(b^2-a^2)r\cos t+2b}{2(b^2-a^2)r\sin t}\sin((b-a)r\sin t)]^2. \qquad (26)
\end{aligned}$$

We see that

$$\begin{aligned}
R \geq & (b^2-a^2)\frac{1}{a}r\sin t-(b^2-a^2)\frac{1}{b}r\sin t\ \exp\{-2(b-a)r\cos t\} \\
& -\exp\{-2(b-a)r\cos t\}2(b^2-a^2)\frac{r\sin t}{b} \\
& \cdot[\frac{(b^2-a^2)r\cos t+2b}{2(b+a)}]^2
\end{aligned}$$

$$
\begin{aligned}
&= (b^2-a^2)r\sin t\ \exp\{-2(b-a)r\cos t\} \\
&\quad \cdot[-\frac{1}{2b(b+a)^2}((b^2-a^2)r\cos t+2b)^2 \\
&\quad +\frac{1}{a}\exp\{2(b-a)r\cos t\}-\frac{1}{b}] \\
&= (b^2-a^2)r\sin t\ \exp\{-2(b-a)r\cos t\} \\
&\quad \cdot[-\frac{1}{2b(b+a)^2}((b^2-a^2)r\cos t+2b)^2-\frac{1}{b} \\
&\quad +\frac{1}{a}(1+\frac{2(b-a)r\cos t}{1!}+\frac{4(b-a)^2r^2\cos^2 t}{2!}) \\
&\quad +\frac{1}{a}(\exp\{2(b-a)r\cos t\} \\
&\qquad -1-\frac{2(b-a)r\cos t}{1!}-\frac{4(b-a)^2r^2\cos^2 t}{2!})].
\end{aligned}
\tag{27}
$$

From the last member of the above equality we see that

$$
\begin{aligned}
&-\frac{1}{2b(b+a)^2}((b^2-a^2)r\cos t+2b)^2-\frac{1}{b} \\
&+\frac{1}{a}(1+\frac{2(b-a)r\cos t}{1!}+\frac{4(b-a)^2r^2\cos^2 t}{2!}) \\
= &\ \frac{b^3-b^2a-ba^2-a^3}{ba(b+a)^2}+\frac{2(b-a)b}{a(b+a)}r\cos t \\
&+\frac{(b-a)^2(4b-a)}{2ba}r^2\cos^2 t.
\end{aligned}
\tag{28}
$$

Therefore under the condition

$$0 < b^3 - b^2a - ba^2 - a^3$$

we see that $R > 0$ if $\sin t$ is positive and $r > 0$ and see that $\Im p(r\exp\{it\}) > 0$, and we conclude that $\zeta(3; s)$ does not vanish on the right half complex plane except for the origin. q.e.d.

Theorem 2. *Assume that the inequality*

$$0 < b^3 - b^2a - ba^2 - a^3$$

holds. Then the probability distribution with density function $g(3; u)$ is infinitely divisible and the probability distribution with density function $f(a, b; x)$ is infinitely divisible.

Proof. By Theorem 1 the Laplace-Stieltjes transformatiom $\zeta(3;s)$ does not vanish on the right half plane except for the origin and we have

$$\begin{aligned}
&-\frac{\zeta'(3;s)}{\zeta(3;s)} \\
= \ & \frac{1}{\sqrt{s}}\Big\{(b^2-a^2)(e^{-2a\sqrt{s}}+e^{-2b\sqrt{s}}) - \frac{2}{\sqrt{s}}(ae^{-2a\sqrt{s}}-be^{-2b\sqrt{s}}) \\
&-\frac{1}{s}(e^{-2a\sqrt{s}}-e^{-2b\sqrt{s}})\Big\} \\
&/\Big\{(b^2-a^2)(\frac{1}{a}e^{-2a\sqrt{s}}+\frac{1}{b}e^{-2b\sqrt{s}}) - \frac{2}{\sqrt{s}}(e^{-2a\sqrt{s}}-be^{-2b\sqrt{s}})\Big\} \\
= \ & \frac{1}{\sqrt{s}}\Big\{(b^2-a^2)(1+e^{-2(b-a)\sqrt{s}}) - \frac{2}{\sqrt{s}}(a-be^{-2(b-a)\sqrt{s}}) \\
&-\frac{1}{s}(1-e^{-2(b-a)\sqrt{s}})\Big\} \\
&/\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{-2(b-a)\sqrt{s}}) - \frac{2}{\sqrt{s}}(1-be^{-2(b-a)\sqrt{s}})\Big\}. \qquad (29)
\end{aligned}$$

From the above we will calculate the inverse Laplace transform,

$$k(t) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\xi-iR_1}^{\xi+iR_1} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds \qquad (30)$$

$$(0<\xi,\ 0<t,\ R_1 = R\cos\epsilon).$$

By the Cauchy theorem

$$\int_\gamma \phi(z)dz = 0,$$

we obtain the following integral in such as a way in the figure after references.

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{A\to B} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds \\
= \ & -\frac{1}{2\pi i}\int_{B\frown C} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds - \frac{1}{2\pi i}\int_{C\frown D} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds \\
&-\frac{1}{2\pi i}\int_{D\to G} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds - \frac{1}{2\pi i}\int_{G\frown H} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds \\
&-\frac{1}{2\pi i}\int_{H\to E} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds - \frac{1}{2\pi i}\int_{E\frown F} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds \\
&-\frac{1}{2\pi i}\int_{F\frown A} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds. \qquad (31)
\end{aligned}$$

(a) The contour integral along the curve $B \frown C$:

$$
\begin{aligned}
&\int_{B\frown C} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds \\
= &\int_{\pi/2-\epsilon}^{\pi} e^{tRe^{i\theta}}\Big[\frac{1}{\sqrt{R}e^{i\theta/2}}\Big\{(b^2-a^2)(1+e^{-2(b-a)\sqrt{R}e^{i\theta/2}}) \\
&-\frac{2e^{-i\theta/2}}{\sqrt{R}}(a-be^{-2(b-a)\sqrt{R}e^{i\theta/2}}) - \frac{e^{-i\theta}}{R}(1-e^{-2(b-a)\sqrt{R}e^{i\theta/2}})\Big\} \\
&/\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{-2(b-a)\sqrt{R}e^{i\theta/2}}) \\
&-\frac{2e^{-i\theta/2}}{\sqrt{R}}(1-e^{-2(b-a)\sqrt{R}e^{i\theta/2}})\Big\}\Big]Re^{i\theta}id\theta \qquad (32)
\end{aligned}
$$

We see that

$$
\begin{aligned}
&\Big|\Big\{(b^2-a^2)(1+e^{-2(b-a)\sqrt{R}e^{i\theta/2}}) \\
&-\frac{2e^{-i\theta/2}}{\sqrt{R}}(a-be^{-2(b-a)\sqrt{R}e^{i\theta/2}}) - \frac{e^{-i\theta}}{R}(1-e^{-2(b-a)\sqrt{R}e^{i\theta/2}})\Big\} \\
&/\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{-2(b-a)\sqrt{R}e^{i\theta/2}}) \\
&-\frac{2e^{-i\theta/2}}{\sqrt{R}}(1-e^{-2(b-a)\sqrt{R}e^{i\theta/2}})\Big\}\Big| \le K \ \ (constant) \qquad (33)
\end{aligned}
$$

for sufficiently large R and we obtain

$$
\begin{aligned}
&\int_{\pi/2-\epsilon}^{\pi} |e^{tRe^{i\theta}}\sqrt{R}e^{i\theta/2}|Kd\theta \\
&\le K\int_{\pi/2-\epsilon}^{\pi/2} e^{tR\cos\theta}\sqrt{R}d\theta \le Ke^{t\xi}\frac{\xi}{\sqrt{R}}\frac{\epsilon}{\sin\epsilon} \to 0 \qquad (34)
\end{aligned}
$$

as $R \to \infty$.

(b) The contour integral along the curve $C \frown D$: On the interval $\pi/2 < \theta < \pi$ we see that

$$
\begin{aligned}
&\Big|\Big\{(b^2-a^2)(1+e^{-2(b-a)\sqrt{R}e^{i\theta/2}}) \\
&-\frac{2e^{-i\theta/2}}{\sqrt{R}}(a-be^{-2(b-a)\sqrt{R}e^{i\theta/2}}) - \frac{e^{-i\theta}}{R}(1-e^{-2(b-a)\sqrt{R}e^{i\theta/2}})\Big\} \\
&/\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{-2(b-a)\sqrt{R}e^{i\theta/2}}) \\
&-\frac{2e^{-i\theta/2}}{\sqrt{R}}(1-e^{-2(b-a)\sqrt{R}e^{i\theta/2}})\Big\}\Big| \le K \ \ (constant) \qquad (35)
\end{aligned}
$$

for sufficiently large R an by the inequality $0 \leq 2\theta/\pi \leq \sin\theta$ for $0 < \theta < \pi/2$ we obtain

$$\begin{aligned}
& \int_{\pi/2}^{\pi} |e^{tRe^{i\theta}} \sqrt{R} e^{i\theta/2}| K d\theta \\
\leq\ & K \int_{\pi/2}^{\pi} e^{tR\cos\theta} \sqrt{R} d\theta \\
=\ & K \int_{0}^{\pi/2} e^{-tR\sin\theta} \sqrt{R} d\theta \\
\leq\ & K \int_{0}^{\pi/2} e^{-2tR\theta/\pi} \sqrt{R} d\theta \\
=\ & K[-\frac{\pi}{2tR} e^{-2tR\theta/\pi} \sqrt{R}]_0^{\pi/2} = K \frac{\pi}{2t\sqrt{R}} (1 - e^{-tR}) \to 0
\end{aligned} \tag{36}$$

as $R \to \infty$.

(c) The contour integral along the curve $D \to G$: We see that

$$\begin{aligned}
& \int_{D \to G} e^{ts} (-1) \frac{\zeta'(3;s)}{\zeta(3;s)} ds \\
=\ & \int_{+R}^{+r} e^{t\rho e^{i\pi}} \Big[\frac{1}{\sqrt{\rho} e^{i\pi/2}} \Big\{ (b^2 - a^2)(1 + e^{-2(b-a)\sqrt{\rho} e^{i\pi/2}}) \\
& -\frac{2e^{-i\pi/2}}{\sqrt{\rho}} (a - be^{-2(b-a)\sqrt{\rho} e^{i\pi/2}}) - \frac{e^{-i\pi}}{\rho} (1 - e^{-2(b-a)\sqrt{\rho} e^{i\pi/2}}) \Big\} \\
& / \Big\{ (b^2 - a^2)(\frac{1}{a} + \frac{1}{b} e^{-2(b-a)\sqrt{\rho} e^{i\pi/2}}) \\
& -\frac{2e^{-i\pi/2}}{\sqrt{\rho}} (1 - e^{-2(b-a)\sqrt{\rho} e^{i\pi/2}}) \Big\} \Big] e^{i\pi} d\rho \\
=\ & \int_{r}^{R} e^{-t\rho} \Big[\Big\{ (b^2 - a^2)(1 + e^{-2(b-a)\sqrt{\rho} i}) \\
& +\frac{2i}{\sqrt{\rho}} (a - be^{-2(b-a)\sqrt{\rho} i}) + \frac{1}{\rho} (1 - e^{-2(b-a)\sqrt{\rho} i}) \Big\} \\
& / \Big\{ (b^2 - a^2)(\frac{1}{a} + \frac{1}{b} e^{-2(b-a)\sqrt{\rho} i}) \\
& +\frac{2i}{\sqrt{\rho}} (1 - e^{-2(b-a)\sqrt{\rho} i}) \Big\} \Big] \frac{d\rho}{\sqrt{\rho} i}.
\end{aligned} \tag{37}$$

(d) The contour integral along the small circle $G \frown H$:

$$\int_{G \frown H} e^{ts} (-1) \frac{\zeta'(3;s)}{\zeta(3;s)} ds$$

$$
\begin{aligned}
= & \int_{\pi}^{-\pi} e^{tre^{i\theta}}\Big[\frac{1}{\sqrt{r}e^{i\theta/2}}\Big\{(b^2-a^2)(e^{-2a\sqrt{r}e^{i\theta/2}}+e^{-2b\sqrt{r}e^{i\theta/2}}) \\
& -\frac{2e^{-i\theta/2}}{\sqrt{r}}(ae^{-2a\sqrt{r}e^{i\theta/2}}-be^{-2b\sqrt{r}e^{i\theta/2}}) \\
& -\frac{e^{-i\theta}}{r}(e^{-2a\sqrt{r}e^{i\theta/2}}-e^{-2b\sqrt{r}e^{i\theta/2}})\Big\} \\
& /\Big\{(b^2-a^2)(\frac{1}{a}e^{-2a\sqrt{r}e^{i\theta/2}}+\frac{1}{b}e^{-2b\sqrt{r}e^{i\theta/2}}) \\
& -\frac{2e^{-i\theta/2}}{\sqrt{r}}(e^{-2a\sqrt{r}e^{i\theta/2}}-e^{-2b\sqrt{r}e^{i\theta/2}})\Big\}\Big]re^{i\theta}id\theta \qquad (38)
\end{aligned}
$$

$\to 0$ as $r \to 0$ since the absolute value of the integrand of the above integral tends to zero from (20) and (21) as r tends to zero.
(e) The contour integral along the curve $H \to E$: We see that

$$
\begin{aligned}
& \int_{H\to E} e^{ts}(-1)\frac{\zeta'(3;s)}{\zeta(3;s)}ds \\
= & \int_{r}^{R} e^{t\rho e^{-i\pi}}\Big[\frac{1}{\sqrt{\rho}e^{-i\pi/2}}\Big\{(b^2-a^2)(1+e^{-2(b-a)\sqrt{\rho}e^{-i\pi/2}}) \\
& -\frac{2e^{i\pi/2}}{\sqrt{\rho}}(a-be^{-2(b-a)\sqrt{\rho}e^{-i\pi/2}}) \\
& -\frac{e^{+i\pi}}{\rho}(1-e^{-2(b-a)\sqrt{\rho}e^{-i\pi/2}})\Big\} \\
& /\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{-2(b-a)\sqrt{\rho}e^{-i\pi/2}}) \\
& -\frac{2e^{i\pi/2}}{\sqrt{\rho}}(1-e^{-2(b-a)\sqrt{\rho}e^{-i\pi/2}})\Big\}\Big]e^{-i\pi}d\rho \\
= & \int_{r}^{R} e^{-t\rho}\Big[\Big\{(b^2-a^2)(1+e^{+2(b-a)\sqrt{\rho}i}) \\
& -\frac{2i}{\sqrt{\rho}}(a-be^{+2(b-a)\sqrt{\rho}i})+\frac{1}{\rho}(1-e^{+2(b-a)\sqrt{\rho}i})\Big\} \\
& /\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{+2(b-a)\sqrt{\rho}i}) \\
& -\frac{2i}{\sqrt{\rho}}(1-e^{+2(b-a)\sqrt{\rho}i})\Big\}\Big]\frac{d\rho}{\sqrt{\rho}i}. \qquad (39)
\end{aligned}
$$

The function $\zeta'(3;s)/\zeta(3;s)$ is symmetric with respect to the real axis and the integrals along the curves $E \frown F$ and $F \frown A$ tend to zero as $R \to \infty$. From the above integrals we obtain

$$k(t)$$

$$
\begin{aligned}
= \lim_{r\to+0,\,R\to+\infty} & \frac{1}{2\pi}\int_r^R e^{-t\rho}\Big[\Big\{(b^2-a^2)(1+e^{-2(b-a)\sqrt{\rho}i}) \\
& +\frac{2i}{\sqrt{\rho}}(a-be^{-2(b-a)\sqrt{\rho}i})+\frac{1}{\rho}(1-e^{-2(b-a)\sqrt{\rho}i})\Big\} \\
& /\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{-2(b-a)\sqrt{\rho}i})+\frac{2i}{\sqrt{\rho}}(1-e^{-2(b-a)\sqrt{\rho}i})\Big\} \\
& +\Big\{(b^2-a^2)(1+e^{+2(b-a)\sqrt{\rho}i}) \\
& -\frac{2i}{\sqrt{\rho}}(a-be^{+2(b-a)\sqrt{\rho}i})+\frac{1}{\rho}(1-e^{+2(b-a)\sqrt{\rho}i})\Big\} \\
& /\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{+2(b-a)\sqrt{\rho}i})-\frac{2i}{\sqrt{\rho}}(1-e^{+2(b-a)\sqrt{\rho}i})\Big\}\Big]\frac{d\rho}{\sqrt{\rho}}. \qquad (40)
\end{aligned}
$$

Let us denote the above $k(t)$ by

$$
k(t)=\int_{+0}^{+\infty}\frac{1}{2\pi}e^{-t\rho}\frac{N}{D}\frac{d\rho}{\sqrt{\rho}}.
$$

We see that

$$
\begin{aligned}
N \;=\; & \Re\Big[\Big\{(b^2-a^2)(1+e^{-2(b-a)\sqrt{\rho}i}) \\
& +\frac{2i}{\sqrt{\rho}}(a-be^{-2(b-a)\sqrt{\rho}i})+\frac{1}{\rho}(1-e^{-2(b-a)\sqrt{\rho}i})\Big\} \\
& \cdot\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{+2(b-a)\sqrt{\rho}i})-\frac{2i}{\sqrt{\rho}}(1-e^{+2(b-a)\sqrt{\rho}i})\Big\} \\
& +\Big\{(b^2-a^2)(1+e^{+2(b-a)\sqrt{\rho}i}) \\
& -\frac{2i}{\sqrt{\rho}}(a-be^{+2(b-a)\sqrt{\rho}i})+\frac{1}{\rho}(1-e^{+2(b-a)\sqrt{\rho}i})\Big\} \\
& \cdot\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{-2(b-a)\sqrt{\rho}i})+\frac{2i}{\sqrt{\rho}}(1-e^{-2(b-a)\sqrt{\rho}i})\Big\}\Big] \\
& =2\Big[\Big\{(b^2-a^2)(1+\cos(2(b-a)\sqrt{\rho})) \\
& -\frac{2b}{\sqrt{\rho}}\sin(2(b-a)\sqrt{\rho}+\frac{1}{\rho}(1-\cos(2(b-a)\sqrt{\rho})\Big\} \\
& \cdot\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}\cos(2(b-a)\sqrt{\rho}))-\frac{2}{\sqrt{\rho}}\sin(2(b-a)\sqrt{\rho})\Big\} \\
& -\Big\{-(b^2-a^2)\sin(2(b-a)\sqrt{\rho}) \\
& +\frac{2}{\sqrt{\rho}}(a-b\cos(2(b-a)\sqrt{\rho}))+\frac{1}{\rho}\sin(2(b-a)\sqrt{\rho})\Big\} \\
& \cdot\Big\{(b^2-a^2)\frac{1}{b}\sin(2(b-a)\sqrt{\rho})-\frac{2}{\sqrt{\rho}}(1-\cos(2(b-a)\sqrt{\rho}))\Big\}\Big]. \quad (41)
\end{aligned}
$$

Let us denote $\sqrt{\rho}$ by y in these calculations. By the calculation we obtain the following form,

$$\begin{aligned} D &= \Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{-2(b-a)yi}) - \frac{2}{yi}(1-e^{-2(b-a)yi})\Big\} \\ &\cdot\Big\{(b^2-a^2)(\frac{1}{a}+\frac{1}{b}e^{+2(b-a)yi}) + \frac{2}{yi}(1-e^{+2(b-a)yi})\Big\} \\ &= \frac{1}{y^2}\Big[(b^2-a^2)^2\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{ab}\cos 2(b-a)y\}y^2 \\ &+8(1-\cos 2(b-a)y) - 4(b^2-a^2)(\frac{1}{a}+\frac{1}{b})y\ \sin 2(b-a)y\Big]. \quad (42) \end{aligned}$$

We see that the denominator D is always positive, i.e., by (27) and (28) we obtain

$$\Big\{(b-a)\frac{(b^3-ab^2-a^2b-a^3)}{ab(a+b)}\Big\}^2 \leq D. \tag{43}$$

We see that the numerator N is always nonnegative. In fact, we obtain

$$N = \frac{2}{aby^2}\{2(a+b)^3((a-b)y\cos((b-a)y)+\sin((b-a)y))^2\}. \tag{44}$$

From the above D and N we see that the value of the integral

$$k(t) = \int_0^\infty \frac{1}{2\pi}e^{-t\rho}\,\frac{N}{D}\frac{d\rho}{\sqrt{\rho}} \tag{45}$$

is positive for all positive t. By the fact that $|N/D| \leq K$ (constant) we obtain that

$$\int_1^\infty \frac{1}{t}k(t)dt \leq \int_1^\infty\{\int_0^\infty \frac{1}{2\pi t}e^{-t\rho}\frac{K}{\sqrt{\rho}}d\rho\}dt = \frac{K}{2\sqrt{\pi}}\int_1^\infty \frac{dt}{t^{3/2}} < \infty,$$

$$\int_0^1 k(t)dt \leq \int_0^1\{\int_0^\infty \frac{1}{2\pi}e^{-t\rho}\frac{K}{\sqrt{\rho}}d\rho\}dt = \frac{K}{2\sqrt{\pi}}\int_0^1 \frac{dt}{t^{1/2}} < \infty.$$

Therefore we see that the probability distribution with density function $g(3:u)$ is infinitely divisible under the condition and the probability distribution with density function of normed product of the 3 dimensional Cauchy densities is infinitely divisible under the condition since it is the variance mixture density of the normal distributions. q.e.d.

If $a=1$ and $b=2$ the condition is satisfied. In fact, we have

$$b^3 - b^2a - ba^2 - a^3 = 1,$$

and it is seen that the probability distribution with density $f(1,2;x)$ is infinitely divisible.

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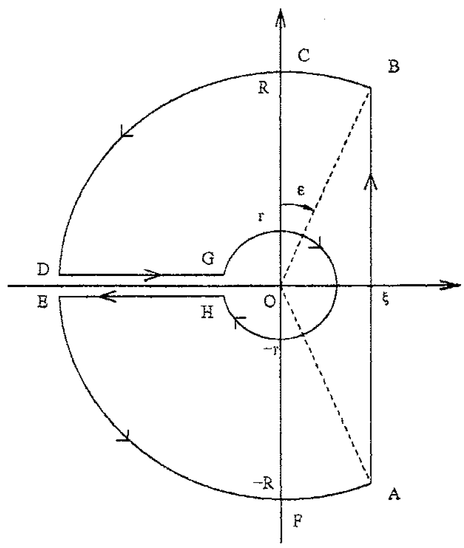
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